HW 7 SOLUTIONS

(1) $$\underbrace{x^2(1-x)^2}_{P(x)}\, y'' + \underbrace{2x}_{Q(x)}\, y' + \underbrace{4}_{R(x)}\, y = 0$$

ZEROS OF $P(x)$ ARE SINGULAR POINTS:

$$P(x) = 0 \quad @ \quad x = 0,\ x = 1$$

EXAMINE:

$$\lim_{x \to x_0} (x - x_0) \frac{Q(x)}{P(x)} \quad \text{AND} \quad \lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)}$$

FOR $x_0 = 0, +1$

$x_0 = 0$:

$$\lim_{x \to 0} x \frac{Q(x)}{P(x)} = \lim_{x \to 0} \frac{x \cdot 2x}{x^2(1-x)^2} = \lim_{x \to 0} \frac{2}{(1-x)^2} = 2 \quad \therefore \text{LIMIT EXISTS} ✓$$

$$\lim_{x \to 0} x^2 \cdot \frac{4}{x^2(1-x)^2} = \lim_{x \to 0} \frac{4}{(1-x)^2} = 4 \quad \text{LIMIT EXISTS} ✓$$

AND

$x_0 = 1$: $$\lim_{x \to 1} (x-1) \frac{2x}{x^2(1-x)^2} = \lim_{x \to 1} \frac{-2x}{x^2(1-x)} \quad \text{LIMIT DOES NOT EXIST} ✗$$

$$\cancel{\lim_{x \to 1} (x-1)^2 \cdot \frac{4}{x^2(1-x)} = \lim_{x \to 1} \frac{-4(x-1)}{x^2} = 0 \quad \text{LIMIT EXISTS}} ✓$$

$\therefore$ $\boxed{x_0 = 0 \text{ IS REGULAR SING. POINT},\ x_0 = 1 \text{ IS IRREGULAR SING. POINT}}$

(2)

$$\underbrace{x(1-x^2)^3}_{P(x)} y'' + \underbrace{(1-x^2)^2}_{Q(x)} y' + \underbrace{2(1+x)}_{R(x)} y = 0$$

$P(x) = x(1-x^2)^3 = 0$ @ $x_0 = 0, \pm 1$

$x_0 = 0$: $\lim_{x \to 0} x^2 \cdot \frac{2(1+x)}{x(1-x^2)^3} = \lim_{x \to 0} \frac{2x(1+x)}{(1-x^2)^3} = 0$

$\lim_{x \to 0} \frac{x \cdot (1-x^2)^2}{x(1-x^2)^3} = \lim_{x \to 0} \frac{1}{1-x} = 1$

$\therefore$ $\boxed{x_0 = 0 \text{ IS REGULAR SINGULAR POINT}}$

$x_0 = 1$: $\lim_{x \to 1} (x-1)^2 \cdot \frac{2(1+x)}{x(1-x^2)^3} = \lim_{x \to 1} \frac{2(1+x)}{x} \cdot \frac{(1-x)^2}{[(1-x)(1+x)]^3}$

$= \lim_{x \to 1} \frac{1}{(1+x)^2} \cdot \frac{2}{x} \cdot \frac{1}{1-x}$ DOES NOT EXIST X

$\boxed{x_0 = 1 \text{ IS IRREGULAR SINGULAR POINT}}$

$x_0 = -1$: $\lim_{x \to -1} \frac{(x+1)(1-x^2)^2}{x(1-x^2)^3} = \lim_{x \to -1} \frac{x+1}{x} \cdot \frac{1}{(1-x)(1+x)} = \lim_{x \to -1} \frac{1}{x(1-x)}$

$= -1/2$

$\lim_{x \to -1} \frac{(x+1)^2(1+x)2}{x(1-x^2)^3} = \lim_{x \to -1} \frac{2}{x(1-x)^3} = -1/4$

$\therefore$ $\boxed{x_0 = -1 \text{ IS REGULAR SING. POINT}}$

(3) $x(3-x)y'' + (x+1)y' - 2y = 0$

SINGULAR POINTS : $x_0 = 0, 3$

$x_0 = 0$: $\lim\limits_{x \to 0} \dfrac{x \cdot (x+1)}{x(3-x)} = \dfrac{1}{3}$, $\lim\limits_{x \to 0} x^2 \cdot \dfrac{-2}{x(3-x)} = 0$

$\therefore$ $\boxed{x_0 = 0 \text{ IS REGULAR SINGULAR POINT}}$

$x_0 = 3$: $\lim\limits_{x \to 3} \dfrac{(x-3)(x+1)}{x(3-x)} = \lim\limits_{x \to 3} \dfrac{-(x+1)}{x} = -4/3$

$\lim\limits_{x \to 3} (x-3)^2\left(\dfrac{-2}{x(3-x)}\right) = \lim\limits_{x \to 3} \dfrac{2(x-3)}{x} = 0$

$\therefore$ $\boxed{x_0 = 3 \text{ IS REGULAR SINGULAR POINT}}$

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(4) $2xy'' + y' + xy = 0$

(a) $\lim\limits_{x \to 0} x \cdot \dfrac{1}{2x} = 1/2$, $\lim\limits_{x \to 0} x^2 \cdot \dfrac{x}{2x} = 0$

$\Rightarrow$ $\boxed{x_0 = 0 \text{ IS REGULAR SINGULAR POINT}}$

(b) GUESS: $y = \sum\limits_{n=0}^{\infty} a_n x^{n+r}$, $y' = \sum\limits_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$

$y'' = \sum\limits_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1}$

(4) (b) (CONTINUED) PLUG GUESS INTO ODE:

$$2\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-1} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} + \underbrace{\sum_{n=0}^{\infty} a_n x^{n+r+1}}_{\text{SHIFT TO GET:}} = 0$$

$$\text{"} \qquad + \qquad \text{"} \qquad + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} = 0$$

$\Rightarrow$

$$2a_0(r)(r-1)x^{r-1} + a_0(r)x^{r-1} + 2a_1(1+r)(r)x^r + a_1(r+1)x^r$$

$$+ \sum_{n=2}^{\infty}\left[2a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2}\right]x^{n+r-1} = 0$$

SET TERMS $= 0$:

$\underline{n=0}$:

$$\left(2a_0 r(r-1) + a_0 r\right)x^{r-1} = 0$$

$$\Rightarrow a_0\left(2r^2 - 2r + r\right) = 0 \qquad a_0 \neq 0$$

$$\Rightarrow \boxed{2r^2 - r = 0} \Rightarrow \boxed{r = 0, \tfrac{1}{2}}$$

INDICIAL EQUATION

ROOTS OF INDICIAL EQ'N

$\underline{n=1}$:

$$\left[2a_1(1+r)(r) + a_1(r+1)\right]x^r = 0$$

$$\Rightarrow a_1\left[2r(r+1) + (r+1)\right] = 0 \qquad \text{BUT } r \text{ MUST } = 0 \text{ OR } \tfrac{1}{2}$$

$$\Rightarrow a_1 = 0.$$

$\underline{n \geq 2}$:

$$\left[a_n\left(2(n+r)(n+r-1) + (n+r)\right) + a_{n-2}\right]x^{n+r-1} = 0$$

$$\Rightarrow \boxed{a_n = \frac{-a_{n-2}}{2(n+r)(n+r-1) + (n+r)}}$$

RECURRENCE RELATION

(4) (c) $r = 1/2$ SOLUTION:

$$a_2 = \frac{-a_0}{2(2+1/2)(1+1/2)+(2+1/2)} = \frac{-a_0}{10}$$

$$a_3 = \frac{-a_1}{(\quad)} = 0 \quad \text{B.C. } a_1 = 0$$

$$a_4 = \frac{-a_2}{2(4+1/2)(3+1/2)+(4+1/2)} = +\frac{1}{10}\cdot a_0 \cdot \frac{1}{(\quad)}$$

$$= \frac{a_0}{360}$$

$$a_5 = 0 \quad \text{B.C. } a_3 = 0.$$

$$a_6 = -\frac{1}{360}a_0 \frac{1}{2(6+1/2)(5+1/2)+(6+1/2)} = -\frac{1}{28{,}080}.$$

$$\Rightarrow y_1 = x^{1/2}\left(a_0 + a_2x^2 + a_4x^4 + a_6x^6 + \dots\right)$$

$$\boxed{y_1 = a_0 x^{1/2}\left(1 - \frac{1}{10}x^2 + \frac{1}{360}x^4 - \frac{1}{28080}x^6 + \dots\right)}$$

(d) $r = 0$ SOLUTION:

$$a_2 = \frac{-a_0}{2(2)(1)+2} = -\frac{1}{6}a_0, \quad a_4 = \frac{-a_2}{2(4)(3)+4} = +\frac{1}{168}a_0$$

$$a_6 = \frac{-a_4}{2(6)(5)+6} = -\frac{1}{11088}a_0$$

$$\Rightarrow \boxed{y_2 = a_0\left(1 - 1/6\,x^2 + 1/168\,x^4 - 1/11088\,x^6 + \dots\right)}$$

(5) $xy'' + y = 0$

(a) $\lim_{x \to 0} x \cdot \frac{0}{x} = 0$, $\lim_{x \to 0} x^2 \cdot \frac{1}{x} = 0$

$\Rightarrow \boxed{x_0 = 0 \text{ IS REG. SING. POINT}}$

(b) $$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1} + \underbrace{\sum_{n=0}^{\infty} a_n x^{n+r}}_{\text{SHIFT}} = 0$$

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

$$\Rightarrow a_0 (r)(r-1) x^{r-1} + \sum_{n=1}^{\infty} \left[a_n (n+r)(n+r-1) + a_{n-1} \right] x^{n+r-1} = 0.$$

SET TERMS $= 0$:

$n=0$: $a_0 r(r-1) x^{r-1} = 0 \Rightarrow$ INDICIAL EQ$^{\underline{n}}$ $\boxed{r(r-1) = 0} \Rightarrow \boxed{r = 0, 1}$ ROOTS OF INDICIAL EQ$^{\underline{n}}$.

$n \geq 1$: $\left[a_n (n+r)(n+r-1) + a_{n-1} \right] x^{n+r-1} = 0 \Rightarrow$

$$\boxed{a_n = \frac{-a_{n-1}}{(n+r)(n+r-1)}}$$ RECURRENCE RELATION

(c) FIND SOL$^{\underline{n}}$ CORRESPONDING TO $r = 1$:

$$a_1 = \frac{-a_0}{(1+1)(1)} = -\frac{a_0}{2}, \quad a_2 = \frac{-a_1}{(2+1)(2)} = +\frac{a_0}{12}$$

$$a_3 = \frac{-a_2}{(3+1)(3)} = \frac{-a_0}{(12)(4)(3)} = -\frac{1}{144} a_0$$

(5) (c) (CONTINUED)

$$y_1 = x^1\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\right)$$

$$\boxed{y_1 = a_0 x\left(1 - \tfrac{1}{2}x + \tfrac{1}{12}x^2 - \tfrac{1}{144}x^3 + \dots\right)}$$

(d) $r_1 = 1$, $r_2 = 0$ $\Rightarrow$ ROOTS DIFFER BY AN INTEGER.

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(6) $3x^2 y'' + 2xy' + x^2 y = 0$

(a) $\lim_{x \to 0} x \cdot \frac{2x}{3x^2} = 2/3$, $\lim_{x \to 0} x^2 \cdot \frac{x^2}{3x^2} = 0$

$\Rightarrow$ $\boxed{x_0 = 0 \text{ IS REGULAR SINGULAR POINT}}$

(b)
$$3\sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r} + 2\sum_{n=0}^{\infty} a_n (n+r)x^{n+r} + \underbrace{\sum_{n=0}^{\infty} a_n x^{n+r+2}}_{\text{SHIFT}} = 0$$

$$\sum_{n=0}^{\infty}\left[3a_n(n+r)(n+r-1) + 2a_n(n+r)\right]x^{n+r} + \sum_{n=2}^{\infty} a_{n-2}x^{n+r} = 0.$$

$$\Rightarrow \left(3r(r-1) + 2r\right)a_0 x^r + \left[3(1+r)(r) + 2(r+1)\right]a_1 x^{r+1}$$
$$+ \sum_{n=2}^{\infty}\left(\left[3(n+r)(n+r-1) + (2(n+r))\right]a_n + a_{n-2}\right)x^{n+r} = 0.$$

(6) (b) SET TERMS EQUAL TO ZERO:

n=0:

$$\left(3r(r-1)+2r\right)a_0 x^r = 0 \Rightarrow \boxed{\left(3r^2-r\right)=0}$$

INDICIAL EQUATION

$$\Rightarrow \boxed{r = 1/3,\, 0}$$ ROOTS

n=1:

$$\left(3(1+r)(r)+2(r+1)\right)a_1 x^{r+1} = 0$$

$$\Rightarrow \left(3r(r+1)+2(r+1)\right)a_1 = 0 \Rightarrow \boxed{a_1 = 0}$$

n≥2:

$$\boxed{\left[3(n+r)(n+r-1)+2(n+r)\right]^{-1}\cdot\left(-a_{n-2}\right) = a_n}$$

RECURRENCE RELATION

(c) $r = 1/3$: $a_1 = a_3 = a_5 = a_7 = \dots = 0$.

$$a_2 = \frac{-a_0}{3(2+1/3)(1+1/3)+2(2+1/3)} = -\frac{a_0}{14}$$

$$a_4 = \frac{-a_2}{3(4+1/3)(3+1/3)+2(4+1/3)} = +\frac{a_0}{728}$$

$$a_6 = \frac{-a_0}{82992} \Rightarrow y_1 = x^{1/3}\left(a_0 + a_2x^2 + a_4x^4 + a_6x^6 + \dots\right)$$

$$\boxed{y_1 = a_0 x^{1/3}\left(1 - 1/14\,x^2 + 1/728\,x^4 - 1/82992\,x^6 + \dots\right)}$$

(6) (d) SOL^{n} FOR $r=0$:

$$a_{n+2} = \frac{-a_n}{3(n)(n-1) + 2n}$$

$$a_2 = -\frac{a}{10}$$

$$a_4 = \frac{+1}{440} a_0$$

$$a_6 = -\frac{1}{44880} a_0 \implies y_2 = x^0\left(a_0 + a_2x^2 + a_4x^4 + a_6x^6 + \dots\right)$$

$$\boxed{y_2 = a_0\left(1 - \frac{1}{10}x^2 + \frac{1}{440}x^4 - \frac{1}{44880}x^6 + \dots\right)}$$

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(7) $xy'' + (1-x)y' - y = 0$

(a) $\lim_{x\to 0} x \cdot \frac{1-x}{x} = 1$, $\lim_{x \to 0} x^2 \cdot \frac{-1}{x} = 0$

$\implies \boxed{x_0 = 0 \text{ IS REG. SING. POINT}}$

(b)
$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-1} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} - \underbrace{\sum_{n=0}^{\infty} a_n(n+r)x^{n+r}}_{\text{SHIFT}}$$

$$-\underbrace{\sum_{n=0}^{\infty} a_n x^{n+r}}_{\text{SHIFT :}} = 0.$$

$$\left[a_0 r(r-1) + a_0 r\right]x^{r-1} + \sum_{n=1}^{\infty}\left[a_n\left((n+r)^2\right) - a_{n-1}(r+n)\right]x^{n+r-1} = 0$$

7 (b) (CONT.)

SET TERMS $= 0$:

$n=0$: $a_0 (r^2) x^{r-1} = 0 \Rightarrow \boxed{r^2 = 0 \text{ INDICIAL EQ}^{\underline{n}}}$

$r = 0$ ROOT OF INDICIAL EQ$^{\underline{n}}$ (WITH MULTIPLICITY 2)

$n \geq 1$: $\left[a_n (n+r)^2 - a_{n-1}(r+n) \right] x^{n+r-1} = 0$

$$\Rightarrow \boxed{a_n = \frac{a_{n-1}(r+n)}{(n+r)^2} = a_{n-1} \frac{1}{n+r}}$$

RECURRENCE RELATION

(c)

$r=0$:

$$a_1 = \frac{a_0 (0+1)}{1^2} = a_0$$

$$a_2 = \frac{1}{2} a_1 = \frac{1}{2} a_0$$

$$a_3 = \frac{1}{3} a_2 = \frac{1}{3 \cdot 2} a_0$$

$$a_4 = \frac{1}{4} \cdot a_3 = \frac{1}{4 \cdot 3 \cdot 2} a_0$$

$\vdots$

$$a_n = \frac{1}{n} \cdot a_{n-1} = \frac{1}{n!} a_0$$

$$\Rightarrow y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$= a_0 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \right)$$

$$= a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x$$

$$\therefore \boxed{y = e^x}$$

(d) ROOT IS REPEATED.